# Between Wqo and Bqo, the Space of Ideals of a Wqo

Yann Pequignot

Joint work with Raphaël Carroy

University of Lausanne

Paris VII

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# Quasiorder

A **quasiorder** (qo) is a set $Q$ together with a *reflexive* and *transitive* binary relation $\leq$. We write $p < q$ for $p \leq q$ and $p \ngeq q$.

1. $Q$ is **well founded** if it admits no infinite $<$-descending chain;
2. $A \subseteq Q$ is an **antichain** if $p \neq q \rightarrow p \nleq q$ for all $p, q \in A$;
3. a sequence $(q_n)_{n \in \omega}$ is called
   - **good** if $\exists m, n \in \omega$ with $m < n$ and $q_m \leq q_n$;
   - **perfect** if $\forall m, n \in \omega$ $m \leq n$ implies $q_m \leq q_n$;
4. $D \subseteq Q$ is a **downset** if $q \in D$ and $p \leq q$ implies $p \in D$. We write $\mathrm{Down}(Q)$ po of downsets of $Q$ with inclusion.
5. for $S \subseteq Q$ we write $\downarrow S = \{p \in Q \mid \exists q \in S \ p \leq q\}$ for the **downward closure** of $S$.
6. give **upset** and **downward closure** the dual meanings.

# Quasiorder

A **well quasiorder** (wqo) is a qo that satisfies one of the following equivalent conditions.

1. $Q$ is well founded and has no infinite antichain;
2. every sequence is good;
3. every sequence admits a perfect subsequence;
4. every upset $U$ admits a finite $F \subseteq Q$ such that $U = \uparrow F$;
5. $(\mathrm{Down}(Q), \subseteq)$ is well founded.

The main tool to show the equivalence is the classical:

**Theorem (Ramsey)**

*Let $k \in \omega$ and let $[\omega]^k = P_0 \cup P_1$ be a partition of the set of sets of natural numbers with cardinality $k$. There exists an infinite $M \subseteq \omega$ such that*

$$\textit{either} \quad [M]^k \subseteq P_0, \quad \textit{or } [M]^k \subseteq P_1.$$

# Closure properties of wqo

Here are examples of wqo's:

- Finite qo's ;
- well ordered set, ordinals;
- any subset of a wqo;
- any quotient of a wqo;
- *finite* products of wqo's;
- *finite* unions of wqo's;

For $s$ and $t$ ordinal sequences in $Q$ we define

$$s \leq_{dom} t \quad \text{iff} \quad \begin{array}{c}\text{there exists a strictly increasing map} \\ h : |s| \to |t| \text{ s.t. } s_i \leq t_{h(i)} \text{ for all } i \in |s|\end{array}$$

**Theorem**

*If $Q$ wqo then the qo $(Q^{<\omega}, \leq_{dom})$ of finite sequences in $Q$ is wqo.*

Wqo's are stable under finite combination. But if $Q$ is wqo

- Is $Q^{\omega}$ wqo?
- And $(\mathrm{Down}(Q), \subseteq)$?
- and $Q^{\mathbf{ON}}$?

# Wqo? Well, we want more

Remember $Q$ is wqo iff $(\text{Down}(Q), \subseteq)$ is well founded.

**Question:**

What is a witness in $Q$ that $\text{Down}(Q)$ is **not** wqo?

- Let $(D_n)_{n\in\omega}$ is a bad (=not good) sequence in $(\text{Down}(Q), \subseteq)$.
- For all $m, n \in \omega$ and all $m < n$: $D_m \nsubseteq D_n$.
- For all $m \in \omega$ build a sequence $(q_{\{m,n\}})_{m<n}$ by choosing

$$q_{\{m,n\}} \in D_m \quad \text{and} \quad q_{\{m,n\}} \notin D_n.$$

- The sequence of sequences $(q_{\{m,n\}})_{m<n}$ satisfies

$$q_{\{m,n\}} \nleq q_{\{n,l\}} \quad \text{for all } m < n < l.$$

otherwise $q_{\{m,n\}} \leq q_{\{n,l\}} \in D_n$ implies $q_{\{m,n\}} \in D_n$.

# Wqo? Well, we want more

**Question:**

What does ensure inside $Q$ that $\mathrm{Down}(Q)$ is wqo?

**Answer:**

Consider sequences of *higher dimension*.

- A **sequence of sequences** is a map $f : [\omega]^2 \to Q$ from the pairs in $\omega$.
- Say a sequence of sequences $f : [\omega]^2 \to Q$ is **good** if

$$\text{there exists } m < n < l \text{ s.t. } f(\{m, n\}) \leq f(\{n, l\}).$$

Recall that: every sequence in $Q$ is good $\leftrightarrow$ $Q$ is wqo.

**Proposition**

*Let $Q$ be a qo. Every sequence of sequences in $Q$ is good* $\leftrightarrow$ $\mathrm{Down}(Q)$ *is wqo.*

# Richard Rado's Example

**Question**

Does there exist a wqo $Q$ such that $\mathrm{Down}(Q)$ is not wqo?

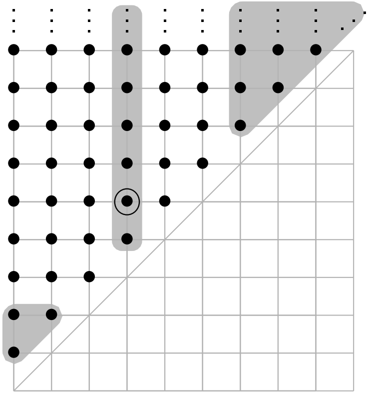
Yes, there is!
Richard Rado
1954

Let $R = ([\omega]^2, \sqsubseteq)$ with

$$\{m, n\} \sqsubseteq \{k, l\}$$

iff

$$\begin{cases} m = k \text{ and } n \leq l, \text{ or} \\ m < n < k < l \end{cases}$$



# Wqo? Well, we want better

We want to define a class of quasiorders such that

- $Q$ is wqo
- Down($Q$) is wqo
- Down(Down($Q$)) is wqo
- Down$^k$($Q$) is wqo
- Down$^\omega$($Q$) is wqo
- Down$^\alpha$($Q$) is wqo

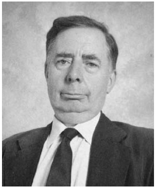
This is done by requiring that

- every sequence is good
- every sequence of sequences is good
- every sequence of sequences of sequences is good
- every sequence of sequences of sequences of sequences... is good
- every ????? is good

We need a transfinite notion of sequence of sequences... supersequences.

# Wqo? Well, we want better

Crispin St J. Nash-Williams:
There is a generalisation
of the classical Ramsey theorem
to the transfinite dimension!



A **barrier** is a family $B$ of finite sets of natural numbers such that

1. $\bigcup B$ is infinite;
2. for all $s, t \in B$, $s \subseteq t$ implies $s = t$;
3. every infinite subset of $\bigcup B$ admits an initial segment in $B$.

**Theorem (Nash-Williams,1965)**

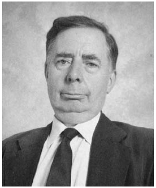
*Let $B$ be a barrier and let $B = P_0 \cup P_1$ be a partition of $B$. Then there exists an infinite $M \subseteq \bigcup B$ such that*

$$\textit{either} \quad B|M \subseteq P_0, \quad \textit{or } B|M \subseteq P_1.$$

*where $B|M = \{s \in B \mid s \subset M\}$.*

# Wqo? Well, we want better

Crispin St J. Nash-Williams:
Wqo? Well, we want better!



For finite set of natural numbers $s$ and $t$ let

$$s \lhd t \quad \text{iff} \quad \begin{array}{c}\text{there exists } u \text{ s.t.}\\ s \sqsubset u \text{ and } t = u \setminus \min u\end{array}$$

- A **supersequence** in $Q$ is a map $f : B \to Q$ from a barrier $B$.
- A supersequence $f : B \to Q$ is **good** if there is $s, t \in B$ with $s \lhd t$ and $f(s) \leq f(t)$.

**Definition (Nash-Williams, 1965)**

A qo $Q$ is a **better quasiorder** (bqo) if every supersequence in $Q$ is good.

# Cauchy sequences and uniform continuity

**Fact**

*Let $(x_n)_{n\in\omega}$ be a sequence in $2^\omega$. The following conditions are equivalent:*

1. $(x_n)_{n\in\omega}$ *is Cauchy;*
2. $\{n \in \omega \mid x_n \in C\}$ *is finite or cofinite for all clopen $C$ of $2^\omega$;*
3. *the map $f : [\omega]^1 \to 2^\omega$, $n \mapsto x_n$ is uniformly continuous.*

Where the barrier $[\omega]^1 = \{\{n\} \mid n \in \omega\}$ is equipped with the uniform structure (metric) inherited by $2^\omega$ via the identification:

$$[\omega]^{<\infty} \longrightarrow 2^\omega$$
$$s = \{2,4,5\} \longmapsto x_s = 001011000\cdots$$

# Cauchy sequences and uniform continuity

**Definition**

Let $f : B \to X$ be a supersequence.

- A **sub-supersequence** of $f$ is a restriction of $f$ to some barrier $B' \subseteq B$.

Remark: sub-supersequences of $f$ are exactly the $f : B|N \to X$ for an infinite $N \subseteq \bigcup B$. Recall $B|N = \{s \in B \mid s \subset N\}$.

**Definition**

For a metric space $X$, say a supersequence $f : B \to X$ is **Cauchy** if it is uniformly continuous when $B$ is equipped with the uniform structure (metric) induced by $2^\omega$.

Every sequence in $2^\omega$ has a Cauchy (convergent) subsequence and

**Theorem (Carroy R. and P.)**

*Every supersequence in $2^\omega$ (i.e. in any 0-dim compact Polish space) has a Cauchy sub-supersequence.*

# Cauchy sequences and uniform continuity

A Cauchy $f : [\omega]^1 \to 2^\omega$ converges and thus extends uniquely to a continuous map

$$\begin{aligned} \overline{f} : \overline{[\omega]^1} &\longrightarrow 2^\omega \\ \{n\} &\longmapsto f(\{n\}) \\ \emptyset = 0^\omega &\longmapsto \lim_n f(\{n\}). \end{aligned}$$

Similarly if $f : B \to 2^\omega$ is Cauchy (i.e. uniformly continuous) then it extends uniquely to a continuous

$$\overline{f} : \overline{B} \longrightarrow 2^\omega$$

### Example

The closure of $[\omega]^2 = [\omega]^{\leq 2}$. A sequence of sequences $f : [\omega]^2 \to X$ in a complete metric space $X$ is Cauchy iff

- for each $n$ we have $f(\{n, m\})_{n<m} \to \overline{f}(\{n\})$, and
- $\overline{f}(\{n\})_{n\in\omega} \to f(\emptyset)$.

# The space of ideals of a wqo

A *non empty* subset $I$ of a qo $Q$ is an **ideal** if

- $I$ is a downset;
- $I$ is directed, i.e. for all $p, q \in I$ there is $r \in I$ with $p \leq r$ and $q \leq r$.

Let $\mathrm{Idl}(Q)$ be the po of ideals of $Q$ under inclusion.

Let $2^Q$ be the generalised Cantor space of subsets of $Q$.

Any qo $Q$ is naturally mapped into $2^Q$ via

$$Q \longrightarrow 2^Q$$
$$q \mapsto \downarrow q.$$

We identify $Q$ (the po quotient of $Q$) with its image in $2^Q$.

**Proposition (M. Pouzet and N. Sauer, 2005)**

*If $Q$ is wqo then the closure of $Q$ in $2^Q$ equals $I(Q)$.*

- *$I(Q)$ is compact;*
- *$I(Q)$ is scattered;*
- *the set of isolated points of $I(Q)$ equals $Q$.*

# Cauchy supersequence in a wqo

**Theorem (Carroy R. and P.)**

*Every supersequence in $2^\omega$ (i.e. in any 0-dim compact Polish space) has a Cauchy sub-supersequence.*

makes essential use of the metrisability of $2^\omega$. However, since

**Proposition**

*Let $Q$ be wqo and $S \subseteq \mathrm{Idl}(Q)$ be countable. Then $\overline{S}$ is countable and metrisable.*

the theorem applies to supersequences in a wqo:

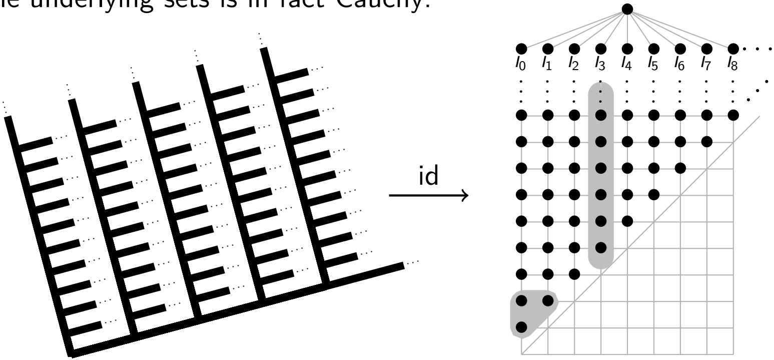
**Corollary (Carroy R. and P.)**

*Every supersequence $g$ in a wqo $Q$ has a Cauchy sub-supersequence $f : B \to Q$. This Cauchy supersequence extends to a continuous*

$$\overline{f} : \overline{B} \to \mathrm{Idl}(Q).$$

# Back to Rado's example

The bad sequence of sequences in $\mathcal{R}$ given by the identity map on the underlying sets is in fact Cauchy:




The barrier $[\omega]^2$ Rado's poset

Its continuous extension restricts to a bad sequence in the non principal ideals:

$$[\omega]^1 \longrightarrow \mathrm{Idl}^*(Q)$$
$$\{n\} \longmapsto I_n$$

# Continuous extensions of supersequences

A point $x$ in a topological space $\mathcal{X}$ is **isolated** if $\{x\}$ is open.
A non isolated point is said to be **limit**.

If $x_n \to x$ in a topological space $\mathcal{X}$ there is $M \in [\omega]^\infty$ such that

either $x$ is isolated and for all $m \in M$ $x_m = x$;

or $x$ is limit and $\begin{cases} \text{either} & x_m \text{ is isolated for all } m \in M; \\ \text{or} & x_m \text{ is limit for all } m \in M. \end{cases}$

For a continuous extension $\overline{f} : \overline{B} \to \mathcal{X}$
of a supersequence $f : B \to \mathcal{X}$ let $\Lambda_{\overline{f}} = \{s \in \overline{B} \mid \overline{f}(s) \text{ is limit}\}$.

**Theorem (Carroy R. and P.)**

*Let $\overline{f} : \overline{B} \to \mathcal{X}$ be a continuous extension of a supersequence $f$ in a topological space $\mathcal{X}$. Then there exists a sub-supersequence $g : B' \to \mathcal{X}$ of $f$ s.t.*

either $\Lambda_{\overline{g}}$ *is empty;*

or $\Lambda_{\overline{g}} = \overline{C}$ *for some barrier $C$.*

# A new proof of a result on bqo

Let $\mathrm{Idl}^*(Q)$ denote the po of non principal ideals of $Q$ under inclusion.
We have $\mathrm{Idl}(Q) = \mathrm{Idl}^*(Q) \cup Q$.

**Theorem (M. Pouzet and N. Sauer, 2005)**

*Let $Q$ be wqo. If $\mathrm{Idl}^*(Q)$ is bqo, then $Q$ is bqo.*

We can give a new topological proof of this result.

# The space of ideals of a wqo

## Last ingredient for the proof

Let $(E_n)_n$ be a sequence in $2^Q$.

$$\bigcap_{n\in\omega} E_n \subseteq \bigcup_{i\in\omega} \bigcap_{j\geq i} E_j \subseteq \bigcap_{i\in\omega} \bigcup_{j\geq i} E_j \subseteq \bigcup_{n\in\omega} E_n.$$

Recall : $E_n \to E$ in $2^Q$ iff $\bigcup_{i\in\omega} \bigcap_{j\geq i} E_j = \bigcap_{i\in\omega} \bigcup_{j\geq i} E_j = E$

The following trick we took in a proof by R. Rado (1954).

### Lemma (Rado's trick)

*Let $Q$ be wqo. For all sequence $(D_n)_{n\in\omega}$ of downsets of $Q$ there exists $M \in [\omega]^\infty$ s.t.*

$$\bigcup_{i\in N} \bigcap_{j\in N/i} I_j = \bigcup_m I_m.$$

### Corollary

*Let $(I_n)_{n\in\omega}$ be a sequence in $\mathrm{Idl}(Q)$. Then there exists an infinite $N \subseteq \omega$ such that $(D_n)_{n\in N}$ converges to $\bigcup_{n\in N} I_n$ in $2^Q$.*

# A new proof of a result on bqo

**Theorem (M. Pouzet and N. Sauer, 2005)**

*Let $Q$ be wqo. If* $\mathrm{Idl}^*(Q)$ *is bqo, then $Q$ is bqo.*

**Sketch of our proof.**

- Let $f : B \to Q$ be a supersequence (to see: $f$ is good);
- Go to a Cauchy sub-supersequence $g : B' \to Q$;
- Extend it continuously to $\overline{g} : \overline{B'} \to \mathrm{Idl}(Q)$;
- Go to a sub-supersequence indexed by $B''$ s.t.

$$\Lambda = \left\{ s \in \overline{B''} \mid f(s) \in \mathrm{Idl}^*(Q) \right\} = \begin{cases} \text{is either empty, or} \\ \overline{C} \text{ for some barrier } C. \end{cases}$$

- three cases:

| | |
|---|---|
| $\Lambda = \emptyset$ | Then $f$ has a constant sub-supersequence. |
| $\Lambda = \overline{C} = \{\emptyset\}$ | $Q$ wqo $\Rightarrow$ $f$ is good. |
| $\Lambda = \overline{C}$ is non trivial | $\mathrm{Idl}^*(Q)$ bqo $\Rightarrow$ $f$ is good. |

□